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SOLUTION CONSTRUCTION OF A PARABOLIC INITIAL-BOUNDARY VALUE PROBLEM FOR DIFFERENT REGRESSION MODELS AT THE LOWER BODY BOUNDARY

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Summary. *A parabolic initial-boundary value problem in a layer with experimental data for the sought function on the lower boundary is formulated and solved. The general solution is specified for different types of regression models.*

The development of approaches and methods for the mathematical description of nonequilibrium processes of various physical nature in artificial and natural objects is driven by the need to create effective methods and estimates for predicting the distribution of technogenic pollution in the environment, controlling the quality of drinking water and improving its purification on an industrial scale, as well as evaluating the impact of aggressive substance diffusion on the reliability and durability of structures [3, 8, 11]. Such researches enable accurate predictions of environmental changes in the environment and promptly implement measures to minimize their negative consequences [12].

However, due to the complexity of the processes and the lack of relevant research, imposing correct boundary conditions on the boundaries often proves to be problematic, even in general cases.

In the paper [4], the solution of the parabolic initial-boundary value problem was statistically investigated using linear regression constructed from experimental data at the lower boundary of the layer, reliable intervals and two-sided critical regions were obtained for both linear regression and the solution of the initial-boundary value problem.

This paper investigates a parabolic initial-boundary value problem that models transport processes, such as heat, mass, and charge transfer, in a layer where experimental data for the sought function are provided on one of the boundaries A regression model is constructed based on this sample, based on which, using a finite integral Fourier transform, the solution of the initial-boundary value problem is found, which is then specified for eight types of regressions.

We consider a physical process that occurs in a layer of thickness x_0, which is described by a parabolic differential equation [2]

$$\frac{\partial w(\tau, x)}{\partial \tau} = B \frac{\partial^2 w(\tau, x)}{\partial x^2}, \qquad (1)$$

where:
$w(\tau, x)$ *is the sought function,* B *is the constant coefficient,* τ *is time,* x *is the spatial coordinate.*

Let the sought function is equal to zero at the initial moment of time:

$$w(\tau, x)\big|_{\tau=0} = 0. \qquad (2)$$

For $\tau > 0$, a constant source acts on the upper surface of the layer:

$$w(\tau, x)\big|_{x=0} = w_* \equiv const. \qquad (3)$$

At the lower boundary of the layer the experimental values of the function $w(\tau, x)$ are known at specific time moments n (Table 1).

Table 1

Experimental data on the sought function at the lower boundary of the layer

τ	τ_1	τ_2	...	τ_i	...	τ_n
$w(\tau)\|_{z_0}$	$w(\tau_1)\|_{z_0}$	$w(\tau_2)\|_{z_0}$	...	$w(\tau_i)\|_{z_0}$	...	$w(\tau_n)\|_{z_0}$

If a regression model is constructed based on the experimental data from Table 1, then the boundary condition on the body surface $x = x_0$ will take the form

$$w(\tau, x)\big|_{x=x_0} = F(\tau). \tag{4}$$

In the following, we limit ourselves to the case $\tau \in [0, \tau_n]$.

To construct the solution of the initial-boundary value problem *(1)-(4)*, we first reduce it to a problem with zero boundary conditions. For this purpose, we use the substitution:

$$v(\tau, x) = w(\tau, x) - w_*\left(1 - \frac{x}{x_0}\right) - F(\tau)\frac{x}{x_0}. \tag{5}$$

Then we get the following problem

$$\frac{\partial v(\tau, x)}{\partial \tau} + \frac{\partial F(\tau)}{\partial \tau}\frac{x}{x_0} = B\frac{\partial^2 v(\tau, x)}{\partial x^2} \tag{6}$$

with initial condition:

$$v(\tau, x)\big|_{\tau=0} = -w_*\left(1 - \frac{x}{x_0}\right) - F(\tau)\big|_{\tau=0}\frac{x}{x_0} \tag{7}$$

and zero boundary conditions :

$$v(\tau, x)\big|_{x=0} = v(\tau, x)\big|_{x=x_0} = 0. \tag{8}$$

Let us apply the finite integral Fourier transform to the initial-boundary value problem *(6)-(8)* [10] ($x \to y_k = k\pi/x_0$, $v(\tau, x) \to \tilde{v}(\tau, y_k)$).Then after transformations we obtain

$$\frac{d\tilde{v}(\tau, y_k)}{d\tau} + By_k^2\tilde{v}(\tau, y_k) = \frac{dF(\tau)}{d\tau}\frac{(-1)^k}{y_k}, \tag{9}$$

$$\tilde{v}(\tau, y_k)\big|_{\tau=0} = -\frac{1}{y_k}\left(w_* + (-1)^{k+1}F(\tau)\right)\Big|_{\tau=0}. \tag{10}$$

The solution of the ordinary differential equation *(9)* using integration by parts is the following expression [5]

$$\tilde{v}(\tau, y_k) = e^{-By_k^2\tau}\left[K + \frac{(-1)^{k+1}}{y_k}\left(F(\tau)e^{By_k^2\tau} - By_k^2\int F(\tau)e^{By_k^2\tau}d\tau\right)\right]. \quad (11)$$

We find the unknown constant K from the initial condition *(10)*:

$$K = \frac{-w_*}{y_k} + 2\frac{(-1)^k}{y_k}F(\tau)\Big|_{\tau=0} - (-1)^\tau By_k\int F(\tau)e^{By_k^2\tau}d\tau\Big|_{\tau=0}$$

and substitute it into the expression *(11)*. Then we have

$$\tilde{v}(\tau, y_k) = -\frac{(-1)^k}{y_k}F(\tau) - e^{-By_k^2\tau}\left[\frac{w_*}{y_k} - 2\frac{(-1)^k}{y_k}F(\tau)\Big|_{\tau=0} + \right.$$

$$\left. + (-1)^k By_k\int F(\tau)e^{By_k^2\tau}d\tau\Big|_{\tau=0} - \frac{(-1)^k}{y_k}By_k^2\int F(\tau)e^{By_k^2\tau}d\tau\right]. \quad (12)$$

After applying the inverse integral Fourier transform to the formula *(12)*, we obtain the solution of the initial-boundary value problem *(6)-(8)*:

$$v(\tau, x) = \frac{-2}{x_0}\sum_{k=1}^{\infty}\left[\frac{(-1)^k}{y_k}F(\tau) + e^{-By_k^2\tau}\left[\frac{w_*}{y_k} - 2\frac{(-1)^k}{y_k}F(\tau)\Big|_{\tau=0} + \right.\right.$$

$$\left.\left. + (-1)^k By_k\int F(\tau)e^{By_k^2\tau}d\tau\Big|_{\tau=0} - \frac{(-1)^k}{y_k}By_k^2\int F(\tau)e^{By_k^2\tau}d\tau\right]\sin(y_k x)\right].$$

After using the formula *(5)*, we obtain the solution of the original initial-boundary value problem *(1)-(4)*

$$w(\tau, x) = w_*\left(1 - \frac{x}{x_0}\right) + F(\tau)\frac{x}{x_0} - \frac{2}{x_0}\sum_{k=1}^{\infty}\left[\frac{(-1)^k}{y_k}F(\tau) + e^{-By_k^2\tau}\left[\frac{w_*}{y_k} - 2\frac{(-1)^k}{y_k}F(\tau)\Big|_{\tau=0} + \right.\right.$$

$$\left.\left. + (-1)^k By_k\int F(\tau)e^{By_k^2\tau}d\tau\Big|_{\tau=0} - \frac{(-1)^k}{y_k}By_k^2\int F(\tau)e^{By_k^2\tau}d\tau\right]\sin(y_k x)\right]. \quad (13)$$

Let us specify the obtained general solution *(13)* for various types of regression function $F(\tau)$ that are often found in mathematical statistics [6, 7, 9].

1°. Let the function $F(\tau)$ be a linear regression, that is $F(\tau) = a\tau + b$, then we have

$$(-1)^k y_k e^{-By_k^2\tau}\int F(\tau)\, e^{By_k^2\tau}\, d\tau = \frac{(-1)^k}{By_k}\left[a\tau + b - \frac{a}{By_k^2}\right];$$

$$(-1)^k y_k e^{-By_k^2\tau}\int F(\tau)\, e^{By_k^2\tau}\, d\tau\Big|_{\tau=0} = \frac{(-1)^k}{By_k}\left[b - \frac{a}{By_k^2}\right];$$

$$w(\tau,x) = w_*\left(1-\frac{x}{x_0}\right) + (a\tau+b)\frac{x}{x_0} - \frac{2}{x_0}\sum_{k=1}^{\infty}\left[\frac{(-1)^k}{y_k}(a\tau+b) + \right.$$

$$\left. + e^{-By_k^2\tau}\left[\frac{w_*}{y_k} - \frac{(-1)^k}{y_k}(2b+a\tau)\right]\right]\sin(y_k x). \qquad (14)$$

The asymptotes of the solution *(14)* have the form

$$w_a^{\pm}(\tau,x) = w_*\left(1-\frac{x}{x_0}\right) + (a\tau+b)\frac{x}{x_0} \pm \frac{2}{x_0}\sum_{k=1}^{\infty}\frac{(-1)^k}{y_k}(a\tau+b).$$

2°. If we have the quadratic regression $F(\tau) = a\tau^2 + b\tau + p$, then we get

$$(-1)^k y_k e^{-By_k^2\tau}\int F(\tau)\, e^{By_k^2\tau}\, d\tau = \frac{(-1)^k}{By_k}\left[a\tau^2 + b\tau + p - \frac{(2a\tau+b)}{By_k^2} + \frac{2a}{B^2 y_k^4}\right];$$

$$(-1)^k y_k e^{-Dy_k^2\tau}\int F(\tau)\, e^{By_k^2\tau}\, d\tau\Big|_{\tau=0} = \frac{(-1)^k}{By_k}\left[p - \frac{b}{By_k^2} + \frac{2a}{B^2 y_k^4}\right];$$

$$w(\tau,x) = w_*\left(1-\frac{x}{x_0}\right) + \left(a\tau^2 + b\tau + p\right)\frac{x}{x_0} - \frac{2}{x_0}\sum_{k=1}^{\infty}\left[\frac{(-1)^k}{y_k}\left(a\tau^2 + b\tau + p\right) + \right.$$

$$\left. + e^{-By_k^2\tau}\left[\frac{w_*}{y_k} - \frac{(-1)^k}{y_k}\left[a\tau^2 + b\tau + 2p - \frac{2a\tau}{By_k^2}\right]\right]\right]\sin(y_k x).$$

For the quadratic regression, the asymptotics of the solution of the initial-boundary value problem can be written as follows

$$w_a^{\pm}(\tau,x) = w_*\left(1-\frac{x}{x_0}\right) + \left(a\tau^2 + b\tau + p\right)\frac{x}{x_0} \pm \frac{2}{x_0}\sum_{k=1}^{\infty}\frac{(-1)^k}{y_k}\left(a\tau^2 + b\tau + p\right).$$

3°. In the general case of polynomial regression $F(\tau) = \sum_{i=0}^{m} a_i \tau^i = P_m(\tau)$ (m is degree of the polynomial) we get

$$(-1)^k y_k e^{-By_k^2\tau}\int F(\tau)\, e^{By_k^2\tau}\, d\tau = \frac{(-1)^k}{By_k}\sum_{i=0}^{m}\frac{(-1)^i}{(By_k^2)^i}\frac{d^i}{d\tau^i}P_m(\tau);$$

$$(-1)^k y_k e^{-Dy_k^2\tau}\int F(\tau)\, e^{By_k^2\tau}\, d\tau\Big|_{\tau=0} = \frac{(-1)^k}{Dy_k}\sum_{i=0}^{m}\frac{(-1)^i}{(Dy_k^2)^i}a_0\,;$$

$$w(\tau,x) = w_*\left(1-\frac{x}{x_0}\right)+\sum_{i=0}^{m}a_i\tau^i\frac{x}{x_0}-\frac{2}{x_0}\sum_{k=1}^{\infty}\left[\frac{(-1)^k}{y_k}\sum_{i=0}^{m}a_i\tau^i+e^{-By_k^2\tau}\left[\frac{w_*}{y_k}-2\frac{(-1)^k}{y_k}a_0+\right.\right.$$

$$\left.\left.+\frac{(-1)^k}{y_k}\sum_{i=0}^{m}\frac{(-1)^i}{(By_k^2)^i}a_i-\frac{(-1)^k}{y_k}\sum_{i=0}^{m}\frac{(-1)^i}{(By_k^2)^i}\frac{d^i}{d\tau^i}P_m(\tau)\right]\sin(y_k x)\right].$$

For the given regression, the asymptotics of the solution of the initial-boundary value problem are as follows:

$$w_a^{\pm}(\tau,x) = w_*\left(1-\frac{x}{x_0}\right)+\sum_{i=0}^{m}a_i\tau^i\frac{x}{x_0}\pm\frac{2}{x_0}\sum_{k=1}^{\infty}\frac{(-1)^k}{y_k}\sum_{i=0}^{m}a_i\tau^i\,.$$

4°. In the case of inverse (hyperbolic) regression $F(\tau) = a + \frac{b}{\tau}$, we have:

$$(-1)^k y_k e^{-By_k^2\tau}\int F(\tau)\, e^{By_k^2\tau}\, d\tau = \frac{(-1)^k a}{By_k}+(-1)^k b y_k Ei(By_k^2\tau)\,;$$

$$(-1)^k y_k e^{-By_k^2\tau}\int F(\tau)\, e^{By_k^2\tau}\, d\tau\Big|_{\tau=0} = \frac{(-1)^k a}{By_k}+(-1)^k b y_k Ei(0)\,;$$

$$w(t,x) = w_*\left(1-\frac{x}{x_0}\right)+\left(a+\frac{b}{\tau}\right)\frac{x}{x_0}-\frac{2}{x_0}\sum_{k=1}^{\infty}\left[\frac{(-1)^k}{y_k}\left(a+\frac{b}{\tau}\right)+\right.$$

$$+e^{-By_k^2\tau}\left[\frac{w_*}{y_k}-2\frac{(-1)^k}{y_k}\lim_{t\to 0}\left(a+\frac{b}{\tau}\right)+\frac{(-1)^k a}{y_k}+\right.$$

$$\left.\left.+(-1)^k bBy_k Ei(0)-\frac{(-1)^k a}{By_k}+(-1)^k bBy_k Ei(By_k^2\tau)\right]\sin(y_k x)\right], \quad (15)$$

where:

$Ei(x) = vp\int_{-\infty}^{x}\frac{e^y}{y}dy$ *is an integral exponential function* [1].

Note that the choice of hyperbolic regression contradicts the imposition of a zero initial condition *(2)*. Therefore, the boundary condition on the surface of the layer $x = x_0$ is written as an integral exponential function

$$w(\tau,x)\big|_{x=x_0} = \begin{cases} 0, & \tau = 0 \\ F(\tau) = a + \dfrac{b}{\tau}, & \tau > 0 \end{cases}.$$

Then the expression for the sought function *(15)* is transformed to the form

$$w(\tau,x) = w_*\left(1-\frac{x}{x_0}\right) + \left(a+\frac{b}{\tau}\right)\frac{x}{x_0} - \frac{2}{x_0}\sum_{k=1}^{\infty}\left[\frac{(-1)^k}{y_k}\left(a+\frac{b}{\tau}\right) + \right.$$
$$+ e^{-By_k^2\tau}\left[\frac{w_*}{y_k} - 2\frac{(-1)^k}{y_k}\left(a+\frac{b}{\tau}\right) + \frac{(-1)^k a}{y_k} + (-1)^k bBy_k Ei(0) - \right.$$
$$\left. - \frac{(-1)^k a}{By_k} + (-1)^k bBy_k Ei(By_k^2\tau)\right]\sin(y_k x), \quad \tau > 0.$$

In this case, the asymptotics of the solution of the initial-boundary value problem take the form

$$w_a^{\pm}(\tau,x) = w_*\left(1-\frac{x}{x_0}\right) + \left(a+\frac{b}{\tau}\right)\frac{x}{x_0} \pm \frac{2}{x_0}\sum_{k=1}^{\infty}\frac{(-1)^k}{y_k}\left(a+\frac{b}{\tau}\right), \quad \tau \neq 0.$$

5°. In the case where the regression is exponential: $F(\tau) = a + be^{p\tau}$, we obtain

$$(-1)^k y_k e^{-By_k^2\tau}\int F(\tau)\, e^{By_k^2\tau}\, d\tau = \frac{(-1)^k a}{By_k} + (-1)^k \frac{by_k}{p+By_k^2} e^{p\tau};$$

$$(-1)^k y_k e^{-Dy_k^2\tau}\int F(\tau)\, e^{By_k^2\tau}\, d\tau\Big|_{\tau=0} = \frac{(-1)^k a}{By_k} + (-1)^k \frac{by_k}{p+By_k^2};$$

$$w(\tau,x) = w_*\left(1-\frac{x}{x_0}\right) + \left(a+be^{p\tau}\right)\frac{x}{x_0} - \frac{2}{x_0}\sum_{k=1}^{\infty}\left[\frac{(-1)^k}{y_k}\left(a+be^{p\tau}\right) + \right.$$
$$\left. + e^{-By_k^2\tau}\left[\frac{w_*}{y_k} - 2\frac{(-1)^k}{y_k}(a+b) + (-1)^k \frac{bBy_k}{p+By_k^2}\left(1-e^{p\tau}\right)\right]\sin(y_k x).\right.$$

For the exponential regression model, the asymptotics of the solution are as follows

$$w_a^{\pm}(\tau,x) = w_*\left(1-\frac{x}{x_0}\right) + \left(a+be^{p\tau}\right)\frac{x}{x_0} \pm \frac{2}{x_0}\sum_{k=1}^{\infty}\frac{(-1)^k}{y_k}\left(a+be^{p\tau}\right).$$

6°. If $F(\tau)=a+b/\sqrt{\tau}$, then

$$(-1)^k y_k e^{-By_k^2\tau}\int F(\tau)\, e^{By_k^2\tau}\, d\tau = \frac{(-1)^k a}{By_k} + (-1)^k b\sqrt{\frac{\pi}{B}}\, e^{-By_k^2\tau} erfi(y_k\sqrt{B\tau});$$

$$(-1)^k y_k e^{-By_k^2\tau}\int F(\tau)\, e^{By_k^2\tau}\, d\tau\Big|_{\tau=0} = \frac{(-1)^k a}{By_k};$$

$$w(t,x) = w_*\left(1-\frac{x}{x_0}\right) + \left(a+\frac{b}{\sqrt{\tau}}\right)\frac{x}{x_0} - \frac{2}{x_0}\sum_{k=1}^{\infty}\left[\frac{(-1)^k}{y_k}\left(a+\frac{b}{\sqrt{\tau}}\right) + \right.$$

$$\left. + e^{-By_k^2\tau}\left[\frac{w_*}{y_k} - 2\frac{(-1)^k}{y_k}\lim_{\tau\to 0}\left(a+\frac{b}{\sqrt{\tau}}\right) - (-1)^k b\sqrt{\pi B}\, e^{-By_k^2\tau} erfi(y_k\sqrt{B\tau})\right]\sin(y_k x)\right], \quad (16)$$

where:

$erfi(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{y^2} dy$ *is a probability integral with an imaginary argument* [1].

For this case of choosing the regression form, it is also necessary to match artificially the conditions at the initial time. Then the boundary condition on the layer surface $x = x_0$ will be written as follows

$$w(\tau,x)\big|_{x=x_0} = \begin{cases} 0, & \tau = 0 \\ a+b/\sqrt{\tau}, & \tau > 0 \end{cases}$$

and the expression for the sought function *(16)* will be transformed to the form

$$w(\tau,x) = w_*\left(1-\frac{x}{x_0}\right) + \left(a+\frac{b}{\sqrt{\tau}}\right)\frac{x}{x_0} - \frac{2}{x_0}\sum_{k=1}^{\infty}\left[\frac{(-1)^k}{y_k}\left(a+\frac{b}{\sqrt{\tau}}\right) + \right.$$

$$\left. + e^{-By_k^2\tau}\left[\frac{w_*}{y_k} - 2\frac{(-1)^k}{y_k}\left(a+\frac{b}{\sqrt{\tau}}\right) - (-1)^k b\sqrt{\pi B}\, e^{-by_k^2\tau} erfi\left(y_k\sqrt{B\tau}\right)\right]\sin(y_k x)\right],$$

$$\tau > 0.$$

The asymptotics of the solution of the initial-boundary value problem can be presented in the form

$$w_a^{\pm}(\tau,x) = w_*\left(1-\frac{x}{x_0}\right) + \left(a+\frac{b}{\sqrt{\tau}}\right)\frac{x}{x_0} \pm \frac{2}{x_0}\sum_{k=1}^{\infty}\frac{(-1)^k}{y_k}\left(a+\frac{b}{\sqrt{\tau}}\right).$$

7°. If $F(\tau)=a+b\sin(p\tau)$, then we obtain

$$(-1)^k y_k e^{-By_k^2\tau}\int F(\tau)\, e^{By_k^2\tau}\, d\tau =$$

$$\frac{(-1)^k a}{By_k} + (-1)^k \frac{by_k}{B^2 y_k^4 + p^2}\left[By_k^2\sin(p\tau) - p\cos(p\tau)\right];$$

$$(-1)^k y_k e^{-By_k^2\tau}\int F(\tau)\, e^{By_k^2\tau}\, d\tau\Big|_{\tau=0} = \frac{(-1)^k a}{By_k} - (-1)^k \frac{bpy_k}{B^2 y_k^4 + p^2};$$

$$w(\tau,x) = c_*\left(1-\frac{x}{x_0}\right) + (a+b\sin(p\tau))\frac{x}{x_0} - \frac{2}{x_0}\sum_{k=1}^{\infty}\left[\frac{(-1)^k}{y_k}(a+b\sin(p\tau)) + \right.$$

$$\left. + e^{-By_k^2\tau}\left[\frac{w_*}{y_k} - 2a\frac{(-1)^k}{y_k} - (-1)^k\frac{bpBy_k}{B^2y_k^4+p^2}\left[1+By_k^2\sin(p\tau) - p\cos(p\tau)\right]\right]\sin(y_k x)\right.\cdot$$

For this case, the asymptotics of the solution of the initial-boundary value problem are as follows:

$$w_a^{\pm}(\tau,x) = c_*\left(1-\frac{x}{x_0}\right) + (a+b\sin(p\tau))\frac{x}{x_0} \pm \frac{2}{x_0}\sum_{k=1}^{\infty}\frac{(-1)^k}{y_k}(a+b\sin(p\tau)).$$

8°. In the case of $F(\tau) = a + b\cos(p\tau)$, we have

$$(-1)^k y_k e^{-By_k^2\tau}\int F(\tau)\, e^{By_k^2\tau}\, d\tau =$$

$$\frac{(-1)^k a}{By_k} + (-1)^k\frac{by_k}{B^2y_k^4+p^2}\left[By_k^2\cos(p\tau) + p\sin(p\tau)\right];$$

$$(-1)^k y_k e^{-By_k^2\tau}\int F(\tau)\, e^{By_k^2\tau}\, d\tau\Big|_{\tau=0} = \frac{(-1)^k a}{By_k} + (-1)^k\frac{bBy_k^3}{B^2y_k^4+p^2};$$

$$w(\tau,x) = w_*\left(1-\frac{x}{x_0}\right) + (a+b\cos(p\tau))\frac{x}{x_0} - \frac{2}{x_0}\sum_{k=1}^{\infty}\left[\frac{(-1)^k}{y_k}(a+b\cos(p\tau)) + \right.$$

$$+ e^{-By_k^2\tau}\left[\frac{w_*}{y_k} - 2(a+b)\frac{(-1)^k}{y_k} + \right.$$

$$\left.\left. + (-1)^k\frac{bBy_k}{B^2y_k^4+p^2}\left[By_k^2 - By_k^2\cos(p\tau) - p\sin(p\tau)\right]\right]\sin(y_k x)\right..$$

In this case, the asymptotics of the solution of the initial-boundary value problem take the following form:

$$w_a^{\pm}(\tau,x) = w_*\left(1-\frac{x}{x_0}\right) + (a+b\cos(p\tau))\frac{x}{x_0} \pm \frac{2}{x_0}\sum_{k=1}^{\infty}\frac{(-1)^k}{y_k}(a+b\cos(p\tau))\cdot$$

Thus, the initial-boundary value problem for the parabolic partial differential equation in a layer was formulated if experimental data of the sought function at certain time moments were known on its lower boundary. First-type boundary conditions were specified at the initial time moment and on the upper surface of the layer. The regression model of general form was constructed based on the sample of experimental data. The solution of the parabolic initial-boundary value problem was obtained using the finite integral Fourier transform. Formulas for solving the initial-boundary value problem were obtained for linear, quadratic, polynomial, hyperbolic, exponential and trigonometric regression models, as well as for regression of the form $F(\tau) = a + b/\sqrt{\tau}$. For the latter type of regression model and for the hyperbolic regression, the boundary and initial conditions are matched. Asymptotic expressions for the solution of the parabolic initial-boundary value problem were provided for all the considered regression models, which are convenient for the statistical evaluation of the obtained solution.

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