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LOCAL AND CONDITIONAL EXTREMUM OF A FUNCTION OF TWO VARIABLES

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The article provides a detailed presentation of algorithms for finding results obtained by J.L. Lagrange. The theory of extrema of functions of many variables, as a part of mathematical analysis, refers to the mathematical foundations of the study of operations. In turn, many optimization problems are actually problems about the conditional extremum of a function of many variables. The relevance of this topic is determined by the fact that when solving modern problems, methods of solving problems for extremum functions of many variables, obtained in the middle of the 18th and the beginning of the 20th centuries, are used. A special place here is occupied by L. Euler and J.L. Lagrange. The purpose of the article is to show the algorithm for finding the conditional and local extremum obtained by J.L. Lagrange.

Local extremum

Let the function $u = f(x, y)$ is defined in the domain D , and the point $M_0(x_0, y_0) \in D$. If there is a neighborhood of a point M_0 that belongs to the area D and for all points of M different from M_0 of this neighborhood, the inequality holds

$$f(M) < f(M_0) \quad (f(M) > f(M_0)),$$

then the point M_0 is called the point of the local maximum (minimum) of the function $f(x, y)$, and the number $f(M_0)$ is called the local maximum (minimum) of this function. The maximum and minimum points of a function are called its extremum points.

Extremum conditions are necessary.

A function $u = f(x, y)$ can have a local extremum only at those points at which the partial derivatives of the first order with respect to the variables x and y are equal to zero or do not exist, i.e. $\frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0$, or do not exist.

Sufficient conditions of extremum.

Let us denote by A, B, C the values of the partial derivatives of the 2nd order $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2}$, at the critical point $M_0(x_0, y_0)$, i.e.

$$A = \frac{\partial^2 u}{\partial x^2}; B = \frac{\partial^2 u}{\partial x \partial y}; C = \frac{\partial^2 u}{\partial y^2}.$$

Then if:

- 1) $\Delta = \begin{vmatrix} A & B \\ B & C \end{vmatrix} > 0$, then $\begin{cases} f(x_0, y_0) = u_{max} & \text{at } A < 0 \\ f(x_0, y_0) = u_{min} & \text{at } A > 0 \end{cases}$
- 2) $\Delta = \begin{vmatrix} A & B \\ B & C \end{vmatrix} < 0$, then there is no extremum;
- 3) $\Delta = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = 0$, then the extremum may or may not be.

The largest and smallest value of the function.

It is known that the function $u = f(x, y)$ is given and continuous in a closed bounded area D , reaches its largest and smallest values in this area. At the internal points of the area, the differential function can acquire these values only at the extremum points, which are according to the rules given above.

To find out the behavior of the function at the boundary of the domain D , it is necessary to use the equation of the boundary (boundary) of the domain D and reduce this problem to finding the absolute extremum of the function of one variable. By comparing the found values of the function inside and on the border of the area, the largest and smallest values are selected in this way.

The largest or smallest value of the function in a given area is called the absolute extremum of the function (respectively, the absolute maximum or absolute minimum of the function) in this area.

The absolute extremum of a function in a given area is reached either at the critical point of the function belonging to this area, or at a point on the boundary of the area.

Example. Find the point of the local extremum and the nature of the function in it $u = 2x^2 - 3xy + 5y^2 + 7$

Find the partial derivatives of the first order of the given function:

$$\frac{\partial u}{\partial x} = 4x - 3y;$$

$$\frac{\partial u}{\partial y} = -3x + 10y.$$

We equate the first derivatives to zero:

$$\begin{cases} -4x + 3y = 0 \\ -3x + 10y = 0 \end{cases} \quad \begin{cases} x = \frac{3}{4}y \\ -3x + 10y = 0 \end{cases}$$

$$-3 \cdot \frac{3}{4}y + 10y = 0;$$

$$-\frac{9}{4}y + 10y = 0;$$

$$-9y + 40y = 0;$$

$$y = 0.$$

$$\begin{cases} x = 0 \\ y = 0 \end{cases}$$

$$\begin{cases} x = 0 \\ y = 0 \end{cases}$$

$$M(0,0)$$

Let's find the partial derivatives of the second order at the point $M(0,0)$:

$$\frac{\partial^2 u}{\partial x^2} = 4;$$

$$\frac{\partial^2 u}{\partial x \partial y} = -3;$$

$$\frac{\partial^2 u}{\partial y^2} = 10.$$

According to Sylvester's criterion, we determine the sign of the quadratic form:

$$\begin{vmatrix} 4 & -3 \\ -3 & 10 \end{vmatrix}$$

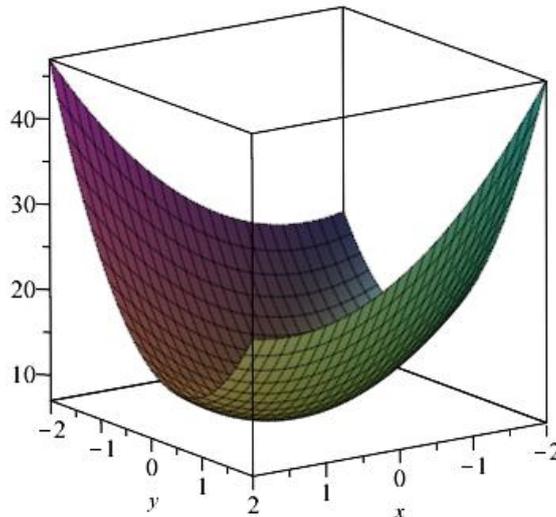
$$\Delta_1 = 4 > 0, \quad \Delta_2 = 31 > 0,$$

The quadratic form is positive definite, the point $M(0,0)$ function has a local minimum.

To visualize the graph of the spatial function around the minimum point, we use the Maple mathematical package.

with(plots);

*plot3d(2 * x^2 - 3 * x * y + 5 * y^2 + 7, x = -2 .. 2, y = -2 .. 2);*



Conditional extremum

The task is to find the extremum $u = f(x, y)$ under the condition that $\varphi(x, y) = 0$.

Variable exclusion method (or substitution method)

From the condition $\varphi(x, y) = 0$, express $y = \psi(x)$ [or $x = \psi(y)$] and substitute $u = f(x, y)$ into the function.

Then the conditional extremum of the given function = the extremum of the function of one variable $u = f(x, \psi(x))$ [or $u = f(y, \psi(y))$].

Algorithm for finding the conditional extremum $f(x, y)$

1 method. Let the function $u = f(x, y)$ be given, for which it is necessary to find extrema under the condition that $\varphi(x, y) = 0$ is the connection equation.

1) The problem of conditional extremum is reduced to finding the usual extremum of the Lagrange function $F(x, y, \lambda) = f(x, y) + \lambda\varphi(x, y)$, λ is the Lagrange multiplier.

First, we write the function $F(x, y, \lambda)$.

2) Next, we compile a system for determining critical points.

To do this, we calculate the partial derivatives of the Lagrange and equate them to zero. The equations of the system are the connection condition

3) The nature of the conditional extremum can be determined by the sign of the second-order differential of the Lagrange function:

$$d^2F = F''_{xx}dx^2 + F''_{xy}dxdy + F''_{yy}dy^2$$

If at a stationary (critical) point $d^2F > 0$, then this is a point of conditional minimum (if at a stationary point $d^2F < 0$, then this is the point of conditional maximum). In practice, the Lagrangian is often denoted by a capital letter $L(x, y, \lambda)$.

2 method.

To prove the positivity of the quadratic form, or the negativity, is sometimes not simple through the coefficients of the mixed derivatives.

You will get this in practice, if the function and the connection condition are selected so that it is difficult to distinguish a complete square under the differential.

Then apply the second method of determining the sign of the second-order differential of the Lagrange function:

From the connection equation we have:

$\varphi'_x dx + \varphi'_y dy = 0$, $dy = -\frac{\varphi'_x}{\varphi'_y} dx$ therefore, at each stationary point is fulfilled:

$$d^2F = F''_{xx}dx^2 + F''_{xy}dxdy + F''_{yy}dy^2 = F''_{xx}dx^2 + 2F''_{xy}dx \left(-\frac{\varphi'_x}{\varphi'_y} dx \right) + F''_{yy} \left(-\frac{\varphi'_x}{\varphi'_y} dx \right)^2$$

$$= \left(-\frac{dx^2}{(\varphi'_y)^2} \right) \cdot \left(-(\varphi'_y)^2 F''_{xx} + 2\varphi'_x \varphi'_y F''_{xy} - (\varphi'_x)^2 F''_{yy} \right)$$

The second factor in quadratic form can be given in the following form

$$H = \begin{vmatrix} 0 & \varphi'_x & \varphi'_y \\ \varphi'_x & F''_{xx} & F''_{xy} \\ \varphi'_y & F''_{xy} & F''_{yy} \end{vmatrix}$$

If $H > 0$, then $d^2F < 0$, then we have a conditional maximum.

If $H < 0$, then $d^2F > 0$, and we have a conditional minimum.

It is often used for the convenience of "playing" signs,

$H > 0$, +, maximum

$H < 0$, -, conditional minimum.

If it is difficult to calculate the determinant, you can write out only the bracket from formula $H = (\varphi'_y)^2 F''_{xx} + 2\varphi'_x \varphi'_y F''_{xy} - (\varphi'_x)^2 F''_{yy}$

We will show how to use the given formulas to find the conditional extremum.

Example. Find the extremum of the function

$$u = x^2 - y + 3, \text{ provided } y - 2x + 5 = 0.$$

1 method.

To do this, express the variable $y(x)$ from the connection condition: $y = 2x - 5$ and substitute in a function $u = x^2 - (2x - 5) + 3$.

Next, we will explore it to the extreme $u' = 2x - 2$.

From the condition $u' = 0$ we obtain

$$2x - 2 = 0, x = 1.$$

$$y(1) = 2 \cdot 1 - 5 = -3.$$

$$M(1, -3) \text{ critical point, value in it } u(1, -3) = 1^2 - (-3) + 3 = 7.$$

To check whether a local maximum or minimum is reached, we substitute any point in the row, in particular at zero, we get $u(0) = 0 - 0 + 5 + 3 = 8$.

The value is greater than at the critical point, so a local minimum is reached at the latter.

That's how quickly, without complicated formulas and calculations, we set the critical point $(1, -3)$ and found out that we have a local minimum there.

Local, because under the condition $y - 2x + 5 = 0$.

The global extremum of the function of two variables $u = x^2 - y + 3$ is different and is searched for on the entire set of definitions, not just on the curve. Now review and compare with Lagrange's method.

1) Add the Lagrange's

$$L(x, y, \lambda) = x^2 - y + 3 + \lambda(y - 2x + 5).$$

2) Find the partial derivatives of L :

$$\frac{\partial L}{\partial x} = 2x - 2\lambda;$$

$$\frac{\partial L}{\partial y} = -1 + \lambda.$$

We write down and solve the system of equations:

$$\begin{cases} 2x - 2\lambda = 0 \\ -1 + \lambda = 0 \\ y - 2x + 5 = 0 \end{cases} \quad \begin{cases} \lambda = 1 \\ x = \lambda = 1 \\ y = 2x - 5 = -3 \end{cases}$$

$$\text{then } u = 1^2 - (-3) + 3 = 7.$$

$(1; -3)$ is a stationary point.

Let's find the second-order partial derivatives of the function $L(x, y, L)$ at a stationary point and substitute them into the differential formula

$$\frac{\partial^2 L}{\partial x^2} = 2;$$

$$\frac{\partial^2 L}{\partial y^2} = 0;$$

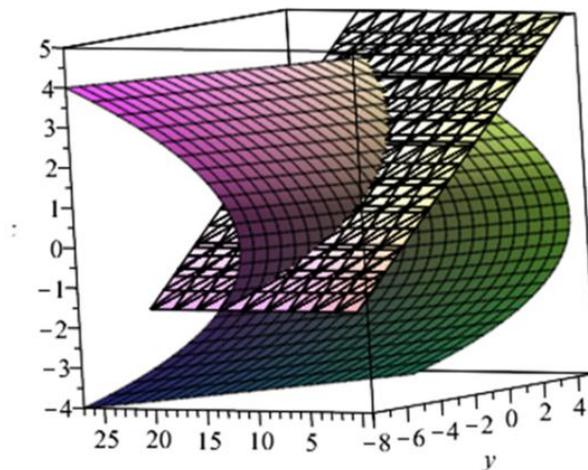
$$\frac{\partial^2 L}{\partial x \partial y} = 0.$$

$$d^2L = \frac{\partial^2 L}{\partial x^2} dx^2 + 2 \frac{\partial^2 L}{\partial x \partial y} dx dy + \frac{\partial^2 L}{\partial y^2} dy^2$$

$$d^2L = 2dx^2 \geq 0$$

Since $d^2L > 0$, the point $(1; 3)$ is the point of the conditional minimum of the function $u = x^2 - y + 3$, i.e. $u_{min} = 7$.

To visualize the z surface and the plane from the connection condition, we will provide a code fragment, how to construct three-dimensional graphs in the Maple package:



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